CONCENTRATION OF STRESSES IN A STRUCTURE MADE OF A UNIDIRECTIONAL MATERIAL

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We consider a structure obtained from a sheet or bar of a unidirectional material by making a number of cuts in it. We use the shear model of [1], which has been widely used for the description of filamentary materials [1-3]. The region of applicability of the model is discussed in detail in [2], and Zweben [4] gives data supporting the good quantitative agreement of the results of calculations based on this model with experimental results. In the present paper we propose an effective method for numerical calculation of the stress–strain state of a structure made of a material containing a large number of filaments.

We assume [1] that in the material the filaments with Young's modulus E are effective only in tension, and the binder with shear modulus G is effective only in shear. Suppose that i is the number of the filament, $i = 1, \ldots, M$; M, total number of filaments; w_i, displacement of filament i along the axis of alignment of the filaments; t, coordinate along this same axis; D and d, thickness of the filaments and the distance between them; E_0, Young's modulus of the filament material ($E = E_0D$). The equilibrium equations have the form

$$E\, w_i'' + (G/d)\,(w_{i-1} - 2w_i + w_{i+1}) = 0, \qquad (1)$$

if the filament is surrounded by the binder on both sides. If between filament i_0 and filament $i_0 + 1$, when $t_1 \leqslant t \leqslant t_2$, there is no binder, then on the left side of (1), for the indicated values of t, we must omit the terms $(G/D)\,(-w_{i_0} + w_{i_0+1})$ from equations i_0 and i_0+1.

We assume that at infinity there is a stress σ applied to the filaments. Subtracting the function σt from the solution, we obtain (preserving the notation mentioned earlier) the conditions: $\bar{w}' \to 0$ as $|t| \to \infty$, $E\, w_{i_l}'(t_l) = -\sigma$, $l = 1, \ldots, L$, where L is the number of cuts in the filaments; $\{(i_l, t_l),\ l = 1, \ldots, L\}$ are the coordinates of these cuts.

It should be noted that a structure of any shape can be obtained by making the appropriate system of cuts in the filaments $\{(i_l, t_l),\ l = 1, \ldots, L\}$ and in the binder $\{(i_m, [t_{1m}, t_{2m}]),\ m = 1, \ldots, N\}$.

After passing to dimensionless coordinates [3], we obtain the problem

$$u_i'' + \Delta u_i = \sum_{m=1}^{N} \delta u_{i_m} \chi_i^m(t); \qquad (2)$$

$$u_{i_l}'(t_l) = -1,\ l = 1, \ldots, L; \qquad (3)$$

$$\bar{u}'(t) \to 0, \qquad |t| \to \infty. \qquad (4)$$

Here $\chi_i^m(t) = 1$ or -1 if $i = i_m$ or $i = i_m + 1$, and $t_{1m} \leqslant t \leqslant t_{2m}$, $\chi_i^m(t) = 0$, otherwise; $\delta u_i = u_{i+1} - u_i$; $\Delta u_i = u_{i-1} - 2u_i + u_{i+1}$.

For a solution of the problem (2)-(4) by ordinary methods, we can obtain the representation:

$$\bar{u}(t) = \sum_{m=1}^{N} \int_{t_{1m}}^{t_{2m}} \bar{G}^m(t-\tau)\, \delta u_{i_m} d\tau + \sum_{k=1}^{L} \sigma_k \bar{U}^k(t), \qquad (5)$$

where $\bar{U}^k(t)$ is a function yielding the displacement as a result of the breaking for $i = i_k$, $t = t_k$ of a single filament; this function was constructed in [3]. The function $\bar{G}^m(t)$ is the solution of the equation

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$$G_i^{m''} + \Delta G_i^m = d_i(t), \tag{6}$$

where $d_{i_m}(t) = \delta(0)$, $d_{i_m+1}(t) = -\delta(0)$, $d_i(t) = 0$ for $i \neq i_m$, $i_m + 1$, with the condition (4). Function $\bar{G}^m(t)$ describes the displacement due to a shear force couple. For numerical calculations it is important that $\bar{U}^k(t)$ and $\bar{G}^m(t)$ can be constructed analytically. This enables us to avoid calculations equivalent to solving a mixed boundary-value problem for a system of M ordinary differential equations.

Let us construct the function $\bar{G}^m(t)$. We seek a solution of (6) by analogy with [3] in the form

$$G_i^m(t) = \sum_{k=1}^{M} (-1)^k C_k^+ \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M} e^{-\lambda_k t}, \; t \geqslant 0,$$

$$G_i^m(t) = \sum_{k=1}^{M} (-1)^k C_k^- \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M} e^{\lambda_k t}, \; t \leqslant 0,$$

$$\lambda_k = \cos(\pi k/2M).$$

For $t \neq 0$, the function $\bar{G}^m(t)$ satisfies Eqs. (6) and the condition (4) [3]. For $t = 0$ we have the splicing condition:

$$G_i^m(+0) = G_i^m(-0); \tag{7}$$

$$G_i^{m'}(+0) = G_i^{m'}(-0), \; i \neq i_m, \; i_m + 1; \tag{8}$$

$$G_{i_m}^{m'}(+0) - G_{i_m}^{m'}(-0) = 1, \; G_{i_m+1}^{m'}(+0) - G_{i_m+1}^{m'}(-0) = -1. \tag{9}$$

We cite a formula from [3]:

$$\sum_{k=1}^{M} q_k \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M} \sin \frac{\pi k\left(i_0 - \frac{1}{2}\right)}{M} = \frac{M}{2} \delta_i^{i_0}, \quad q_k = 1, \; k < M, \; q_M = 1/2. \tag{10}$$

From (7), making use of (10), we obtain $C_i^+ = C_i^-$. Then from (8) we have the equation

$$\sum_{k=1}^{M} \lambda_k C_k^+ \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M} = 0, \; i \neq i_m, \; i_m + 1,$$

and solving this for $\lambda_k C_k^+$, we obtain on the basis of (10)

$$\lambda_k C_k^+ = q_k \left(p \sin \frac{\pi k\left(i_m - \frac{1}{2}\right)}{M} + q \sin \frac{\pi k\left(i_m + \frac{1}{2}\right)}{M} \right), \tag{11}$$

$\forall p, q \in R$. Since $\lambda_M = 0$, it follows that coefficient C_M^+ describing the displacement of the structure as a rigid body is indeterminate, and for solvability of (11), its right side must vanish when $k = M$, and hence $p = q$. Substituting the resulting expression for $\lambda_k C_k^+$ into (9) we have $pM = (-1)^{i_m+1}$, from which we finally find that

$$G_i^m(t) = \sum_{k=1}^{M} (-1)^{i+i_m+1} \frac{q_k}{M} \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M} \sin \frac{\pi k i_m}{M} e^{-\lambda_k |t|}. \tag{12}$$

In an analogous manner, we can construct $\bar{U}^k(t)$ and $\bar{G}^m(t)$ for the three-dimensional case. The function $\bar{U}^k(t)$ for this case was given in [5]. The function $\bar{G}(t)$, describing a couple applied to filaments (i_0, j_0) and $(i_0, j_0 + 1)$, has the form

$$G_{ij}(t) = \sum_{k,m=1}^{M_1 M_2} (-1)^{i_0+j_0+i+j} \frac{2}{M_1 M_2} q_k q_m \sin \frac{\pi k\left(i - \frac{1}{2}\right)}{M_1} \sin \frac{\pi m\left(j - \frac{1}{2}\right)}{M_2} \frac{\lambda_m}{\lambda_{km}} \sin \frac{\pi m j_0}{M_2} e^{-\lambda_{km}|t|}, \tag{13}$$

TABLE 1

$L/2$ \ l	2	4	8	10
10	2,41 1,18	2,56 1,33	2,64 1,42	2,69 1,45
14	2,81 1,51	2,97 1,60	3,02 1,71	3,12 1,74

where M_1, M_2 are the numbers of filaments in the first and second direction; $\lambda_{km} = \sqrt{\lambda_k^2 + \lambda_m^2}$. In what follows, for the sake of simplicity, we shall consider the three-dimensional case a one-dimentional bar, i.e., a layer: $M_1 = 1$, $M_2 = M$. All the results can be carried over to the three dimensional case by simply replacing the influence functions for a layer by the influence functions for a bar.

Thus, the kernels $\bar{G}^m(t)$ and $\bar{U}^k(t)$ in (5) are known, and we can obtain equations for $\{\delta u_{i_m}, \sigma_k\}$. In the cracks, applying the operator δ to (5), we have

$$\delta u_{i_m} = \sum_{n=1}^{N} \int \delta G_{i_m}^n \delta u_{i_n} + \sum_{k=1}^{L} \sigma_k \delta U_{i_m}^k. \tag{14}$$

The second group of equations is obtained from the conditions along the cuts in the filaments:

$$\sum_{n=1}^{N} \int G_{i_l}^{n'}(t_l - \tau)\, \delta u_{i_n} d\tau + \sum_{k=1}^{L} \sigma_k U_{i_l}^{k'}(t_l) = -1. \tag{15}$$

Equations (14) and (15) can be rewritten as an equation of the second kind;

$$\begin{gathered} \delta u_{i_m} = \sum_{n=1}^{N} \int \delta G_{i_m}^n \delta u_{i_n} + \sum_{k=1}^{L} \sigma_k \delta U_{i_m}^k, \\ \sigma_l = -\sum_{n=1}^{N} \int G_{i_l}^{n'}(t_l - \tau)\, \delta u_{i_n} d\tau + \sum_{k=1}^{L} \left[\delta_l^k - U_{i_l}^{k'}(t_l)\right] \sigma_k - 1. \end{gathered} \tag{16}$$

The operator on the right side of (16), $R: L_2^N \times R^L \to L_2^N \times R^L$, is compact, since $\delta\bar{G}^m$, $\bar{G}^{m'} \in C(R)$. Then, according to the Fredholm alternative, Eqs. (16) are solvable, since the solution is unique. After solving (16), we can reconstruct the function $\bar{U}(t)$ everywhere in accordance with (5). From (5) it follows that $\bar{u} \in C^1(R)$ (except at the points $\{t_k\}_{k=1}^L \cup \{t_{1n}\}_{n=1}^N \cup \{t_{2n}\}_{n=1}^N$).

The solution of system (14), (15) was carried out numerically as follows: the integral part of the system (14), (15) was discretized with a step T, after which we obtained a system of algebraic equations of dimension $(P + L) \times (P + L)$, where $P = \sum_{m=1}^{N} (t_{2m} - t_{1m})/T + N$ is the number of nodes in the discretization and L is the number of cuts in the filaments. The solution was carried out on the BESM-6 computer. The program is the same for an arbitrary number of cuts. Let us illustrate the application of the proposed method by showing the results of the calculations for two types of problems.

We shall give the results of calculations for a cutout of rectangular shape. In this problem we selected an acceptable discretization step T. It was found that the step may be taken from the interval [0.2–0.05] (dimensionless variables). If the step in this interval was halved, the corresponding solutions differed by no more than 1–0.5%.

We found that the concentration of stresses and the shear deformations are localized in a region of the order of 2–3 $\max(\sqrt{E_0 dD/G}, \mathscr{L})$, where $\mathscr{L}$ is the characteristic dimension of the cutout. In this region the stress concentrations may considerably (200% or more) exceed the average overload on the filaments, $K_{av} = M/(M - L/2)$, where M is the total number of filaments in the structure and L is the number of cut filaments. An analogous result was found for other shapes of cutouts (circles, triangular notches). The presence of the quantity $\sqrt{E_0 dD/G}$, corresponding to the dimensionless unit of length, in the formula determining the

TABLE 2

l/S / S		0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	m
0,6	γ	2,67	2,66	2,66	2,68	2,72	2,89	2,89	2,99	3,11	3,27	1,005
0,8	γ	2,26	2,25	2,22	2,23	2,25	2,31	—	—	—	—	1,015
1	γ	1,97	1,92	1,89	1,87	1,89	1,93	2,00	2,11	2,27	2,50	1,05
	$K4$	1,75	1,64	1,51	1,36	1,19	1,00	0,80	0,56	0,30	0,00	
	$K5$	1,76	1,71	1,65	1,61	1,56	1,52	1,48	1,43	1,37	1,31	
1,5	γ	1,52	1,42	1,35	1,31	1,30	1,33	1,39	1,51	1,69	1,69	1,17
2	γ	1,26	1,13	1,03	0,98	0,96	0,98	1,04	1,16	1,35	1,66	1,32
	$K4$	1,94	1,79	1,61	1,42	1,22	1,02	0,81	0,59	0,33	0,00	
	$K5$	1,97	1,92	1,89	1,87	1,85	1,84	1,83	1,81	1,79	1,76	
3	γ	0,96	0,78	0,67	0,60	0,58	0,60	0,63	—	—	—	1,60
5	γ	0,67	0,47	0,35	0,30	0,28	0,29	0,34	0,45	0,66	—	2,39

dimensions of the area of localization of the disturbance constitutes one of the features of the behavior of the composite material in question which were clearly determined in the process of numerical calculation.

Table 1 gives the values of the maximum coefficient of stress concentration (in the upper part of the cell) and shear concentration (in the lower part of the cell) for a rectangular cutout. In the calculations the total number of filaments was M = 50; the length l of the cutout and the width L/2 (in the filaments) are shown in Table 1. The maximum stress and shear concentrations in all the calculated cases were found in the filament adjacent to the edge of the cutout for $t = \pm l/2$ (the corners of the rectangular cutout).

Let us consider the strength of this structure. For this we revert to the dimensional coordinate z and the displacement $\overline{w}$:

$$z = (E_0 dD/G)^{1/2} t, \quad \overline{w} = (d\sigma^2/E_0 GD)^{1/2} \overline{u}.$$

Suppose that we are given the condition that the filaments and the binder will fail at the stresses: $E_0 w_i' = \sigma^*$, $G w_i = \tau^*$. Then the tensile load σ at which the failure of the filament (σ_f) or the binder (σ_b) begins will be equal, respectively, to

$$\sigma_f = D\sigma^*/\varepsilon, \quad \sigma_b = (E_0 D/Gd)^{1/2}(\tau^*/\gamma). \tag{17}$$

Suppose that we are given the condition that failure occurs at the deformations: $w_i' = \varepsilon^*$, $\delta w_i = \gamma^*$. Then

$$\sigma_f = D\varepsilon^*/E_0\varepsilon, \quad \sigma_b = (E_0 GD/d)^{1/2}(\gamma^*/\gamma) \tag{18}$$

(ε and γ are the maximum stress concentration and the maximum dimensionless shear (from Table 1)).

As can be seen from (17) and (18), σ_f is not explicitly dependent in either case on the characteristics G and d of the binder. It is true that these quantities appear indirectly in (17) and (18), determining the dimensionless length of the cutout, but the dependence of ε and γ on this length is rather weak, and, as can be shown, it decreases at an exponential rate as the dimensionless length increases. From the second formulas in (17) and (18) it follows that a decrease in the quantity d/D leads to an increase in the "shearing" strength of the structure (subject to the same conditions concerning the length of the cutout).

For a cutout of small width, 2ε, we can obtain a solution by a different method. We write system (16) in the form

$$f_i = \sum_{j=1}^{N} \int \delta G_{ij} f_j + \sum_{k=1}^{L} \sigma_k \delta U_i^k; \tag{19}$$

$$\sum_{k=1}^{L} \sigma_k U_{i_l}^{k'} = - \sum_{j=1}^{N} \int G'_{i_l j} f_j - 1. \tag{20}$$

The norm of the integral operator in (19) is less than unity [6], and therefore

$$\bar{f} = \sum_{k=1}^{L} \sigma_k \sum_{n=0}^{\infty} \varepsilon^n R^n \delta \bar{U}^k, \tag{21}$$

where $R : \bar{f} \in C(R) \to \frac{1}{\varepsilon} \sum_{j=1}^{N} \int_{-\varepsilon}^{\varepsilon} \delta G_{ij} f_j.$

Substituting (21) into (20), taking account of the fact that $\sum_{k=1}^{L} \sigma_k \delta U_i^k(0) = 0,$ because of the symmetry of the cutout, we find that $\bar{f}(0) = 0$. Then, by (20),

$$\sum_{k=1}^{L} \sigma_k U_{i_l}^{k'}(t_l) = -1 + O(\varepsilon^2). \tag{22}$$

This formula describes (to within ε^2) two series of cuts in the filaments spaced at distances of 2ε. For this problem we can easily obtain an approximate solution. Taking account of the symmetry of the cutout, $\sigma_k = \sigma_{k+L/2}$, L is the total number of cuts in the filaments. The function $\bar{U}^k(t)$ has the form [3]

$$U_i^k(t) = \sum_{l=1}^{M} A_{il}^k \left[-e^{\lambda_l (t-\varepsilon)} + e^{-\lambda_l (t+\varepsilon)} \right],$$

where the coefficients A_{il}^k are given in [3]. We expand the exponents in a series and substitute the resulting expressions into (22). Taking account of the equilibrium equations and the fact that $U_i^k(0) = 0$ when $k \neq i$, we find

$$\sum_{k=1}^{L/2} \sigma_k \left[\sum_{l=1}^{M} A_{il}^k \lambda_l - \varepsilon U_k^k(0) \right] = -\frac{1}{2} + O(\varepsilon^2). \tag{23}$$

Then, denoting by σ_k^0 the solution of a problem of the form (22) for one series of filaments, we have from (23)

$$\sigma_k = \frac{1}{2} \left[\sigma_k^0 + \varepsilon \sum_{l=1}^{L/2} a_{kl} U_l^l(0) \sigma_l^0 \right] + O(\varepsilon^2),$$

where $(a_{ik}) = \left(\sum_{l=1}^{M} A_{il}^k \lambda_l \right)^{-1}$. After this the solution is determined from the formula

$$\bar{u}'(t) = \sum_{k=1}^{L/2} \sigma_k \bar{U}^{k'}(t) + O(\varepsilon^2). \tag{24}$$

It follows from the numerical calculations that:

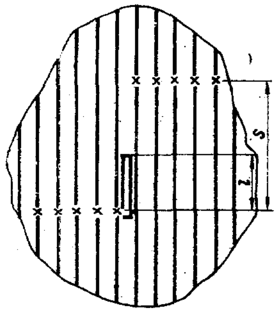

Fig. 1

1) the quantities $U_k^k(0)$ are practically independent of k (their oscillations do not exceed 1%), and we can take $U_k^k(0) = 0.25$;

2) for the matrices $\left(\sum_{l=1}^{M} A_{il}^{k}\lambda_l\right) = (b_{ik})$ we have $b_{kk} \approx -0.63,\ b_{k+1,k} \approx b_{k,k+1} \approx 0.20$, and there is further attenuation as we move away from the main diagonal. Taking $(b_{ik}) \approx -0.63E$, we obtain form (24) an approximate formula for calculating the stress concentration at the edge of a thin rectangular cutout

$$K(t) \approx K_\sigma - 0.4\varepsilon(K_\sigma - 1), \tag{25}$$

where $K_\sigma > 1$ is the coefficient of concentration of the stresses which arise when we cut one series of L/2 filaments. As can be seen, the stresses at the edge of the cutout are independent of t (to within ε^2) and decrease as the width of the cutout increases. The numerical calculation confirms that the stresses at the edge of the cutout are independent of t for small values of ε. From the results of the calculations we can recommend formula (25) for use in the interval $\varepsilon \leqslant 1/2$ (in dimensionless variables).

A second problem solved by means of the indicated program was the problem of the propagation of a shear crack between two series of filament breaks spaced at a distance of S (see Fig. 1). In the case of one series of breaks, the problem was considered in [7], which revealed the stable nature of the propagation of the crack, i.e., σ_b, defined by the second equations in (17) and (18), increases with increasing crack length l (and vice versa).

Table 2 shows the values of $\delta u_i(l) = \gamma$, $\gamma = \sigma_b^{-1}$ (to within the constant factors in (17) and (18)). In the calculations the total number of filaments is M = 50. The data in Table 2 are given for the case of L/2 = 5 breaks in each series. In both cases we give: the coefficient of stress concentration at the end of the crack (in the filament adjacent to the crack on the left, K4) and at the beginning of the crack (in the filament adjacent to the crack on the right, K5).

As can be seen from Table 2 and the second equations in (17) and (18), in this case the stable development of the crack takes place only up to a certain crack length l^*, which, according to calculations, may be estimated at $l^* = (0.3\text{–}0.5)S$. After the crack reaches this length, its further growth does not require any increase of the load at infinity. If for a given load $\sigma < \sigma_b \equiv \sigma^*$ the length of the crack $l(\sigma) < l^*$, then the ratio σ^*/σ, where σ^* : $(\sigma^*) = l^*$, yields the relative value of the additional load required for a jump of the crack to the second series of filament breaks. For a smaller load, the crack is in a stable stage and there is no jump. We introduce the number $m = \sigma^*/\sigma(0) = \gamma(0)/\gamma^*$, where γ^* corresponds to σ^*, l^*; $\sigma(0)$, $\gamma(0)$ correspond to a crack of "zero" length ($\gamma(0)$ is taken from the first column of Table 2). It can be seen from Table 2 that $\sigma^*/\sigma \leq m$ and the number m is the maximum possible margin of relative additional load (hereinafter called simply the additional load). As can be seen, for $S \leqslant 1$ (in dimensionless variables the quantity corresponding to unity is $\sqrt{E_0 dD/G}$) the value m of the additional load is close to unity. This means that the crack in the binder between series of breaks which are relatively close to each other ($S \leqslant 1$) can in practice have only two states: either a fully developed crack (a jump to the second series of breaks) or complete absence of a crack. In the case of longer cracks the additional load becomes markedly greater than unity, and the shear crack, as can be seen from the indicated values of K4 and K5, retards the failure of the filaments. The possible additional load on the structure in the general case must be determined with due regard to the strength of the filaments as well.

The crack behaved similarly for different numbers of broken filaments: $L/2 = 10$, $L/2 = 15$. In these cases too, we had stable development of the crack only so long as its length did not exceed $(0.3-0.5)S$. As the number of broken filaments increased, the additional load decreased slightly: for $S = 1$, we had $m = 1.05$ when $L/2 = 5$, $m = 1.03$ when $L/2 = 10$, and $m = 1.025$ when $L/2 = 15$; for $S = 2$, the values of m were 1.32, 1.26, and 1.24, respectively. The above-described manner of propagation of a shear crack is, of course, only one of the possible ways of propagation of cracks in a unidirectional material. In specific cases we must consider all possible variants of failure. The program described above enables us to carry out such calculations within an acceptable time.

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